Siegel-Veech transform and counting problems

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November 18, 2025

This is the report of the author's reading project conducted in FS 2025 in University of Zurich under the supervision of Prof. Ulcigrai. The main goal of this report is to describe the almost everywhere counting result of Eskin and Masur from a dynamical point of view.

§1 Introduction

A *translation surface* S is a pair (X, ω) where X is a Riemann surface and ω is a holomorphic 1-form. The terminology is motivated by the fact that integrating ω (away from its zeros) gives an atlas of charts to $\mathbb{C}$ whose transition maps are translations.

A *saddle connection* γ on S is a geodesic segment connecting two zeros of ω with none in its interior. Associated to each saddle connection is its *holonomy vector*.

$$z_\gamma = \int_\gamma \omega \in \mathbb{C} \simeq \mathbb{R}^2.$$

Its length is given by

$$|z_\gamma| = \int_\gamma |\omega|.$$

We denote the set of holonomy vectors by Λ_ω, which is a countable discrete subset of $\mathbb{C} \simeq \mathbb{R}^2$.

The moduli space Ω_g of genus g translation surfaces is the vector bundle over the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces, with fiber over each Riemann surface X given by $\Omega(X)$, the vector space of holomorphic 1-forms on X. Ω_g decomposes into strata depending on the combinatorics of the differentials.

As a consequence of Riemann-Roch theorem, the orders of the zeros of ω must sum to $2g-2$, there is a stratum associated to each integer partition of $2g-2$. Each of these strata has at most three connected components.

The flat metric associated to a one-form ω also gives a notion of area on the surface. We consider the subset of area 1 surfaces of a connected component of a stratum, and denote it by $\mathcal{H}$. By abuse of notation, we will often simply refer to this as a stratum, and we will denote elements of it by (X, ω).

The group $GL^+(2, \mathbb{R})$ acts on Ω_g via linear post-composition with charts, preserving the combinatorics of differentials. The subgroup $SL(2, \mathbb{R})$ preserves each area 1 subset, so it acts on each stratum $\mathcal{H}$. On each stratum, there is a natural measure μ, known as the *Masur–Veech* (or Lebesgue) measure, constructed using period coordinates on strata. A crucial result, independently shown by W. Veech and the H. Masur, is that μ is a finite $SL(2, \mathbb{R})$-invariant ergodic measure on each stratum $\mathcal{H}$.

§2 Siegel-Veech transform

We now define the *Siegel–Veech transform*. Fix a stratum $\mathcal{H}$. Let $B_c(\mathbb{R}^2)$ denote the space of bounded, compactly supported functions on $\mathbb{R}^2$. For $f \in B_c(\mathbb{R}^2)$, we define its Siegel–Veech transform, denoted by $\widehat{f}$, as a function on the stratum $\mathcal{H}$ by

$$\widehat{f}(X,\omega) = \sum_{v \in \Lambda_\omega} f(v).$$

This sum is finite for any fixed f and (X,ω), since Λ_ω is discrete and f has compact support. For example, if $f = \chi_{B(0,R)}$, the characteristic function of the Euclidean ball of radius R centered at the origin, then

$$\widehat{\chi}_{B(0,R)}(X,\omega) = \sum_{v \in \Lambda_\omega} \chi_{B(0,R)}(v) = \#\big(\Lambda_\omega \cap B(0,R)\big).$$

Theorem 2.1

Let ρ be the normalized Masur–Veech measure on the stratum $\mathcal{H}$. Then for any $f \in B_c(\mathbb{R}^2)$, there exists a constant $0 < c_{\mathcal{H}} < \infty$ such that

$$\int_{\mathcal{H}} \widehat{f}\, d\rho = c_{\mathcal{H}} \int_{\mathbb{R}^2} f(x,y)\, dxdy. \tag{1}$$

To establish this result, one first shows that $\widehat{f} \in L^1(\mathcal{H},\rho)$, ensuring the finiteness of the left-hand side. This technical fact was proved by Veech [**V98**], and more recently, Athreya, Cheung, and Masur [**L^2**] showed that $\widehat{f} \in L^2(\mathcal{H},\rho)$.

Proof. As discussed before, we can define a positive linear functional Φ on $C_c(\mathbb{R}^2)$ by

$$\Phi(f) = \int_{\mathcal{H}} \widehat{f}\, d\rho.$$

By the Riesz representation theorem, there exists a Radon measure λ on $\mathbb{R}^2$ such that

$$\int_{\mathcal{H}} \widehat{f}\, d\rho = \Phi(f) = \int_{\mathbb{R}^2} f\, d\lambda.$$

Since ρ is $SL(2,\mathbb{R})$-invariant, the measure λ must also be $SL(2,\mathbb{R})$-invariant. We claim that

$$\lambda = a\delta_0 + bm$$

for some $a, b \in \mathbb{R}_{\geq 0}$, where δ_0 is the Dirac measure at the origin and m is Lebesgue measure on $\mathbb{R}^2 \setminus \{0\}$. Assuming the claim, we have

$$\Phi(f) = af(0) + b\int_{\mathbb{R}^2\setminus\{0\}} f\, dm, \quad f \in C_c(\mathbb{R}^2).$$

It remains to show that $a = 0$. Consider the indicator functions $f = \chi_{B(0,r)}$ as $r \to 0$. Both sides of the equation tend to 0, since every translation surface has a positive lower bound on the length of its shortest saddle connection. Hence $a = 0$, which proves the theorem.

Now we prove the claim. Recall that $SL(2,\mathbb{R})$ acts transitively on $\mathbb{R}^2 \setminus \{0\}$ and fixes the origin. If λ had an atom outside the origin, then by transitivity and $SL(2,\mathbb{R})$-invariance, every point in $\mathbb{R}^2 \setminus \{0\}$ would have positive measure. This contradicts the fact that λ is Radon. Thus the only possible atom of λ is at the origin, so we can write $\lambda = a\delta_0 + \lambda|_{\mathbb{R}^2\setminus\{0\}}$. It remains to determine $\lambda|_{\mathbb{R}^2\setminus\{0\}}$. Note that

$$\mathrm{Stab}_{SL(2,\mathbb{R})}(e_1) = N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Hence we can write $\mathbb{R}^2 \setminus \{0\}$ as the homogeneous space $SL(2,\mathbb{R})/N$. By Weil's theorem (see, for example, [**I25**] for a detailed exposition), there is a unique $SL(2,\mathbb{R})$-invariant measure on $SL(2,\mathbb{R})/N$, and hence on $\mathbb{R}^2 \setminus \{0\}$, which is Lebesgue measure. This shows that $\lambda|_{\mathbb{R}^2\setminus\{0\}} = bm$ for some $b \geq 0$. □

The Siegel–Veech transform was introduced by Veech [**V98**] to study counting problems. Given a translation surface (X, ω), let

$$N(\omega, R) = \#\big(\Lambda_\omega \cap B(0,R)\big)$$

denote the number of saddle connections of length at most R. Masur [**M90**] proved that there exist constants

$$0 < c_1(X,\omega) \leq c_2(X,\omega)$$

such that

$$c_1(X,\omega) \leq \frac{N(\omega,R)}{R^2} \leq c_2(X,\omega).$$

As discussed above, we have $\widehat{\chi}_{B(0,R)} = N(\cdot, R)$. The Siegel–Veech transform thus computes the mean of $N(\cdot, R)$:

$$\int_{\mathcal{H}} N(\omega,R)\, d\rho = c_{\mathcal{H}} \pi R^2.$$

It is natural to ask how one can understand the asymptotic behavior of $N(\omega, R)$ for a generic translation surface (X, ω). This was established by Eskin and Masur:

Theorem 2.2 ([**EM01**])
For ρ-almost every $\omega \in \mathcal{H}$,

$$\lim_{R\to+\infty} \frac{N(\omega,R)}{\pi R^2} = c_{\mathcal{H}}.$$

Remark 2.3 The above theorem does not immediately imply a corresponding result for Veech surfaces or billiards, since the set of such surfaces has measure zero. However, as we shall see in Section 5, the counting problem in the case of Veech surfaces can be reduced to counting the orbit points of the associated Veech groups.

§3 From counting to equidistribution

The next two sections mainly follow the excellent exposition in [**AM24**]. This section is the core of the note: we introduce the *triangle trick*, which allows us to transform the problem of counting holonomy vectors into a dynamical problem concerning the $SL(2,\mathbb{R})$-action on stratum (more precisely, an ergodic average along an $SL(2,\mathbb{R})$-orbit).

For $t \in \mathbb{R}$ and $\theta \in [0, 2\pi)$, let

$$g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \quad r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The main object here is the averaging operator: if X is a space with an $SL(2,\mathbb{R})$-action and $h : X \to \mathbb{C}$, define

$$A_t h(x) = \frac{1}{2\pi} \int_0^{2\pi} h(g_t r_\theta x)\, d\theta.$$

For our purposes, X will be either $\mathcal{H}$ or $\mathbb{R}^2$, and the Siegel–Veech transform introduced in the previous section provides the link between these two settings.

We now explain how this operator is related to counting holonomy vectors. Let $\mathcal{T}$ be the triangle with vertices $(0,0), (1,1), (-1,1)$ in $\mathbb{R}^2$. Observe that the action of g_{-t} stretches $\mathcal{T}$ vertically and contracts it horizontally. More precisely, $g_{-t}\mathcal{T}$ is a tall, thin triangle with vertices $(0,0), (e^{-t/2}, e^{t/2}), (-e^{-t/2}, e^{t/2})$, whose cone angle at the origin is

$$\psi_t = 2\arctan(e^{-t}).$$

For $v \in \mathbb{R}^2$, we have

$$(A_t \chi_{\mathcal{T}})(v) = \frac{1}{2\pi} \int_0^{2\pi} \chi_{\mathcal{T}}(g_t r_\theta v)\, d\theta = \frac{1}{2\pi} \left| \{\theta \in [0, 2\pi) : r_\theta v \in g_{-t}\mathcal{T}\} \right|,$$

where $|\cdot|$ denotes Lebesgue measure on $[0, 2\pi)$. It is easy to see that for fixed t, the value of $(A_t\chi_{\mathcal{T}})(v)$ depends only on $\|v\|$. More precisely,

$$(A_t \chi_{\mathcal{T}})(v) = \begin{cases} 0, & \|v\| \geq \sqrt{e^t + e^{-t}}, \\ \dfrac{\psi_t - 2\arccos\left(\frac{e^{t/2}}{\|v\|}\right)}{2\pi}, & e^{t/2} < \|v\| < \sqrt{e^t + e^{-t}}, \\ \dfrac{\psi_t}{2\pi}, & \|v\| \leq e^{t/2}. \end{cases}$$

Thus, for any $v \in \mathbb{R}^2$,

$$\frac{\psi_t}{2\pi} \chi_{B(0, e^{t/2})}(v) \ \leq \ A_t(\chi_{\mathcal{T}})(v) \ \leq \ \frac{\psi_t}{2\pi} \chi_{B(0, \sqrt{e^t + e^{-t}})}(v).$$

Summing over holonomy vectors $v \in \Lambda_\omega$, we obtain for all $t \in \mathbb{R}$,

$$\frac{\psi_t}{2\pi} N(\omega, e^{t/2}) \ \leq \ \widehat{A_t(\chi_{\mathcal{T}})}(\omega) \ \leq \ \frac{\psi_t}{2\pi} N(\omega, \sqrt{e^t + e^{-t}}). \tag{2}$$

Note that

$$\log\left(\sqrt{e^t + e^{-t}}\right) = \log\left(e^{t/2}\sqrt{1 + e^{-2t}}\right) = \tfrac{1}{2}\left(t + \log(1 + e^{-2t})\right).$$

Setting $s(t) = t + \log(1 + e^{-2t})$, we deduce

$$\frac{1}{\psi_t} \widehat{A_t(\chi_{\mathcal{T}})}(\omega) \ \leq \ \frac{1}{2\pi} N(\omega, e^{s(t)/2}).$$

Multiplying both sides by $\psi_{s(t)}$ and applying (2) to $N(\omega, e^{s(t)/2})$, we obtain

$$\frac{\psi_{s(t)}}{\psi_t} \widehat{A_t(\chi_{\mathcal{T}})}(\omega) \ \leq \ \frac{\psi_{s(t)}}{2\pi} N(\omega, e^{s(t)/2}) \ \leq \ \widehat{A_{s(t)}(\chi_{\mathcal{T}})}(\omega).$$

Since

$$\lim_{t\to+\infty}\frac{\psi_{s(t)}}{\psi_t}=\lim_{t\to+\infty}\frac{2e^t}{\psi_t}=1,$$

we conclude that, writing $R=e^{s(t)/2}$,

$$\lim_{R\to+\infty}\frac{N(\omega,R)}{\pi R^2}=\lim_{t\to+\infty}\widehat{A_t(\chi_{\mathcal{T}})}(\omega),$$

provided the limit exists. Finally, since $\chi_{\mathcal{T}}$ is compactly supported, the sum defining $\widehat{\chi_{\mathcal{T}}}$ is finite, and hence

$$\widehat{A_t(\chi_{\mathcal{T}})}(\omega)=A_t(\widehat{\chi_{\mathcal{T}}})(\omega)=\frac{1}{2\pi}\int_0^{2\pi}\widehat{\chi_{\mathcal{T}}}(g_t r_\theta\omega)\,d\theta.$$

In conclusion, we have reduced the problem of understanding the asymptotic behavior of $N(\omega,R)$ to analyzing the limiting behavior of $A_t(\widehat{\chi_{\mathcal{T}}})(\omega)$, which is an ergodic average of $\widehat{\chi_{\mathcal{T}}}$ along an $SL(2,\mathbb{R})$-orbit.

§4 Equidistribution of large circles

As shown in the previous section, we need to prove for ρ-almost every $\omega\in\mathcal{H}$, the limit of the following ergodic average exists:

$$\lim_{t\to\infty}A_t(\widehat{\chi_{\mathcal{T}}})(\omega)=c_{\mathcal{H}}\int_{\mathbb{R}^2}\chi_{\mathcal{T}}=c_{\mathcal{H}}.$$

Theorem 4.1

Let $\varphi\in C_c(\mathbb{R}^2)$. Then for ρ-almost every $\omega\in\mathcal{H}$,

$$\lim_{t\to\infty}A_t(\widehat{\varphi})(\omega)=\int_{\mathcal{H}}\widehat{\varphi}\,d\rho=c_{\mathcal{H}}\int_{\mathbb{R}^2}\varphi(x,y)\,dxdy. \tag{3}$$

The main ergodic tool we use is Nevo's pointwise ergodic theorem for the operators A_t. To state the theorem we first need the definition of K-finite function.

Definition 4.2. Let X be either $\mathcal{H}$ or $\mathbb{R}^2$. A function f on X is called *K-finite* if the linear span of the functions $\{f\circ r_\theta:\theta\in[0,2\pi]\}$ is finite dimensional. Equivalently, f is K-finite if and only if there exists an integer m such that for all $x\in X$, $p(\theta)=f(r_\theta x)$ is a trigonometric polynomial of degree at most m.

To get some idea of this definition, one can notice that the charateristic functions of the rotation invariant sets, for example $\chi_{B(0,R)}$ are K-finite functions (since they are K-invariant) on $\mathbb{R}^2$.

Theorem 4.3 (Nevo)

Let X be either $\mathcal{H}$ or $\mathbb{R}^2$ and let μ be an ergodic $SL(2,\mathbb{R})$ invariant probability measure on X. Suppose $\phi\in C_c(\mathbb{R})$ is a non-negative bump function of unit integral, and suppose $f\in L^{1+\kappa}(X,\mu)$ for some $\kappa>0$ is a K-finite function. Then, for μ-almost every x in X,

$$\lim_{\tau\to\infty}\int_{-\infty}^{\infty}\phi(\tau-t)A_t(f)(x)dt=\int_X f\,d\mu. \tag{4}$$

The goal of this section is to sketch how to apply Nevo's ergodic theorem to our setting, clearly we need to carefully handle at least two things: the 'convolution'-like formula on the left-hand side of (4) and the K-finite functions. We start with this convolution. In fact, we can rewrite it as the following:

$$\int_{-\infty}^{\infty} \phi(\tau - t) A_t(f)(x)\, dt = \int_{-\infty}^{\infty} \phi(s) A_{\tau-s}(f)(x)\, ds = \int_{-\infty}^{\infty} \phi(s) A_\tau(f)(g_{-s}x)\, ds.$$

Then given $\eta \in C_c^\infty(\mathbb{R})$ a non-negative bump function of unit integral, $f \in C_c(\mathbb{R}^2)$ we define our convolution as

$$\eta * f(v) = \int_{-\infty}^{+\infty} \eta(t) f(g_{-t}v)\, dt.$$

The difference of it from the usual convolution is that it is an operation between a funtion on $\mathbb{R}$ and a function on $\mathbb{R}^2$. But it also has the good approximation property as the usual convolution.

Lemma 4.4

Fix $\eta \in C_c^\infty(\mathbb{R})$ a non-negative bump function of unit integral supported in $[-1,1]$. For $\kappa > 0$, define the rescaling

$$\eta_\kappa(t) = \kappa^{-1}\eta(t/\kappa).$$

whose support is in $[-\kappa, \kappa]$. Then for $f \in C_c(\mathbb{R}^2)$, $\eta_\kappa * f$ converges uniformly to f as $\kappa \to 0$.

Proof. Since f is continuous with compact support, it is uniformly continuous: given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(g_{-t}v) - f(v)| < \varepsilon \qquad \text{whenever } |t| < \delta,\ v \in \mathbb{R}^2.$$

Take $0 < \kappa < \delta$. For any $v \in \mathbb{R}^2$ we have

$$\big|(\eta_\kappa * f)(v) - f(v)\big| = \left|\int_{-\infty}^{+\infty} \eta_\kappa(t)\big(f(g_{-t}v) - f(v)\big)\, dt\right| \le \int_{-\kappa}^{\kappa} \eta_\kappa(t)\, |f(g_t v) - f(v)|\, dt.$$

By the choice $\kappa < \delta$ the integral is bounded by ε, hence

$$|(\eta_\kappa * f)(v) - f(v)| \le \varepsilon \int_{-\kappa}^{\kappa} \eta_\kappa(t)\, dt = \varepsilon.$$

Since the bound is uniform in v, this shows $\|\eta_\kappa * f - f\|_\infty \le \varepsilon$. As $\varepsilon > 0$ was arbitrary, $\eta_\kappa * f \to f$ uniformly as $\kappa \to 0$. □

Lemma 4.5

The set of K-finite functions is dense in $C_c(\mathbb{R}^2)$. Moreover, the Siegel-Veech transform of a K-finite function is K-finite.

Proof. Given $R > 0$, consider the following family $\mathcal{F} \subset C(\overline{B(0,R)})$ of functions:

$$\mathcal{F} = \{f_{m,n} : m, n \in \mathbb{Z}\},$$

where for $v = (r\cos\alpha, r\sin\alpha) \in \mathbb{R}^2$, $f_{m,n}(v) = r^m e^{in\alpha} \in \mathbb{C}$. Notice that

$$f_{m,n}(r_\theta v) = r^m e^{in(\alpha+\theta)} = e^{in\theta} f_{m,n}(v).$$

In another word, $\mathrm{span}_{\mathbb{C}}\{f_{m,n} \circ r_\theta : \theta \in [0,2\pi]\} \subset \mathbb{C} f_{m,n}$. Hence for every $m,n \in \mathbb{Z}$, $f_{m,n}$ is K-finite.

Now, we want to use the Stone-Weierstrass theorem to show that the linear span of $\mathcal{F}$ over $\mathbb{C}$ (which is clearly also a set of K-finite functions) is actually dense in $C(\overline{B(0,R)})$. First, observe that

$$f_{m_1,n_1} \cdot f_{m_2,n_2} = f_{m_1+m_2,n_1+n_2} \in \mathcal{F}, \ \overline{f_{m,n}} = f_{m,-n} \in \mathcal{F}.$$

Therefore, $\mathcal{A} = \mathrm{span}_{\mathbb{C}}\mathcal{F} \subset C(\overline{B(0,R)})$ is an algebra consists of K-finite functions. One can also easily show that it is nowhere vanishing and seperating points and hence we can apply Stone-Weierstrass theorem to imply $\mathrm{span}_{\mathbb{C}}\mathcal{F}$ is dense.

Now let $\varphi \in C_c(\mathbb{R}^2)$ be a K-finite function. For $\theta \in [0,2\pi)$,

$$\widehat{\varphi} \circ r_\theta(\omega) = \sum_{v \in \Lambda_{r_\theta \omega}} \varphi(v) = \sum_{v \in \Lambda_\omega} \varphi \circ r_\theta(v).$$

Since φ is K-finite and the summation in the Siegel Veech transform is finite, we have $\widehat{\varphi}$ is K-finite. □

We are now prepared to apply Nevo's theorem to establish the existence of the ergodic average. While certain technical details—such as the specific choice of parameters—will be omitted, our focus will be on conveying the main idea of the proof.

Proof of theorem 4.1. Fix $\varepsilon > 0$. Choose $l > 0$ such that $\mathrm{supp}\,\varphi \subseteq H = \overline{B(0,l)}$, and set $H_1 = \overline{B(0,l+1)}$. Note that $\varphi \leq \|\varphi\|\chi_H$.

Suppose first that $\varphi \in C_c(\mathbb{R}^2)$ is K-finite. By Lemma 4.5, its Siegel-Veech transform $\widehat{\varphi}$ is also K-finite. Hence, by Nevo's theorem, there exists $T > 0$ such that for all $\tau > T$,

$$\left| \int_{-\infty}^{\infty} \phi(s) A_\tau(\widehat{\varphi})(g_{-s}\omega)\, ds - \int_{\mathcal{H}} \widehat{\varphi}\, d\rho \right| < \varepsilon,$$

for ρ-almost every ω, where $\phi \in C_c(\mathbb{R})$ is a nonnegative bump function of unit integral. A key observation is that

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(s) A_\tau(\widehat{\varphi})(g_{-s}\omega)\, ds &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} \sum_{v \in \Lambda_\omega} \phi(s)\varphi(g_{\tau-s} r_\theta v)\, d\theta\, ds \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{v \in \Lambda_\omega} \phi * \varphi(g_t r_\theta v)\, d\theta \\
&= A_\tau \widehat{\phi * \varphi}(\omega).
\end{aligned}$$

Let η be as in Lemma 4.4. Choose $\kappa_0 > 0$ such that for all $\kappa < \kappa_0$,

$$|(\eta_\kappa * \varphi)(v) - \varphi(v)| < \varepsilon, \quad v \in H,$$

and $g_{-\kappa} H \subseteq H_1$. The latter implies $\chi_H \leq \eta_\kappa * \chi_{H_1}$. Since χ_{H_1} is K-finite (indeed K-invariant), enlarging T if necessary we obtain

$$A_\tau(\widehat{\eta_\kappa * \chi_{H_1}})(\omega) < \int_{\mathcal{H}} \widehat{\chi_{H_1}}\, d\rho + \varepsilon = c_{\mathcal{H}} m(H_1) + \varepsilon, \quad \tau > T.$$

Combining these estimates, for all $\tau > T$ and $\kappa < \kappa_0$ we have

$$\begin{aligned}
\left| A_\tau(\widehat{\varphi}) - \int_{\mathcal{H}} \widehat{\varphi}\, d\rho \right| &\leq \left| A_\tau(\widehat{\varphi} - \widehat{\eta_\kappa * \varphi}) \right| + \left| A_\tau(\widehat{\eta_\kappa * \varphi}) - \int_{\mathcal{H}} \varphi\, d\rho \right| \\
&< A_\tau(\varepsilon \widehat{\chi_H}) + \varepsilon \\
&\leq \varepsilon A_\tau(\widehat{\eta_\kappa * \chi_{H_1}}) + \varepsilon \\
&< \varepsilon(c_{\mathcal{H}} m(H_1) + \varepsilon) + \varepsilon.
\end{aligned}$$

for ρ-almost every ω. This completes the proof for K-finite functions.

Now consider a general $\varphi \in C_c(\mathbb{R}^2)$. Lemma 4.5 provides a sequence of K-finite functions $\{f_n\}$ supported in H, such that for some $n_0 \in \mathbb{N}$ and all $n > n_0$,

$$|\varphi - f_n| \leq \varepsilon \chi_H.$$

Then

$$\left| A_\tau(\widehat{\varphi})(\omega) - A_\tau(\widehat{f_n})(\omega) \right| \leq \varepsilon A_\tau(\widehat{\chi_H}) \leq \varepsilon(c_{\mathcal{H}} m(H_1) + \varepsilon),$$

and

$$\left| \int_{\mathcal{H}} \widehat{\varphi}\, d\rho - \int_{\mathcal{H}} \widehat{f_n}\, d\rho \right| \leq c_{\mathcal{H}} \int_{\mathbb{R}^2} |\varphi - f_n| \leq \varepsilon c_{\mathcal{H}} m(H).$$

Therefore,

$$\left| A_\tau(\widehat{\varphi})(\omega) - \int_{\mathcal{H}} \widehat{\varphi}\, d\rho \right| \leq \left| A_\tau(\widehat{f_n})(\omega) - \int_{\mathcal{H}} \widehat{f_n}\, d\rho \right| + C(\mathcal{H}, \varphi)\varepsilon.$$

Since we have prove the theorem for K-finite function, letting $\tau \to \infty$ and then $n \to \infty$, we deduce that the same convergence holds for φ.

The proof is complete.

□

However, the proof of Theorem 2.2 is not yet complete, since $\chi_{\mathcal{T}} \notin C_c(\mathbb{R}^2)$. Of course, one can approximate it by some $f \in C_c(\mathbb{R}^2)$ such that

$$\left| \int_{\mathbb{R}^2} (\chi_{\mathcal{T}} - f) \right|$$

is small, and consequently

$$\left| \int_{\mathcal{H}} (\widehat{\chi_{\mathcal{T}}} - \widehat{f})\, d\rho \right| = c_{\mathcal{H}} \left| \int_{\mathbb{R}^2} (\chi_{\mathcal{T}} - f) \right|$$

is also small. The difficulty lies in the fact that, since $\chi_{\mathcal{T}}$ is discontinuous, the difference $f - \chi_{\mathcal{T}}$ necessarily exhibits a jump along $\partial\mathcal{T}$. As a result, one cannot obtain an inequality of the form

$$|f - \chi_{\mathcal{T}}| \leq \varepsilon \chi_{\overline{B(0,R)}},$$

which played a crucial role in the proof of Theorem 4.1 in controlling the difference of Siegel-Veech transform under A_τ. In the present setting, a more refined understanding of the geometry of the stratum is required. To be more specific, we need to control the volume/measure of the stratum that consists of the surfaces that contains a lot of short saddle connections. Due to the limitations of the author's knowledge, we refer the reader to [**EM01**; **EMZ03**; **AFM23**] for further details.

Proposition 4.6

For all $\varepsilon > 0$, there exists a function $f_\varepsilon \in C_c(\mathbb{R}^2)$ and $T > 0$ so that for all $t \geq T$ and ρ almost every $\omega \in \mathcal{H}$,

$$\left| A_\tau(\widehat{f_\varepsilon} - \widehat{\chi_{\mathcal{T}}})(\omega) \right| < \varepsilon \quad \text{and} \quad \left| \int_{\mathcal{H}} (\widehat{\chi_{\mathcal{T}}} - \widehat{f})\, d\rho \right| < \varepsilon.$$

Using this, we can easily finish the proof of theorem 2.2.

§5 Counting in Veech surfaces

We end this note with a brief discussion on counting in Veech surfaces. Given $(X,\omega) \in \mathcal{H}$, recall that the *Veech group* $\Gamma(\omega)$ of (X,ω) is the stabilizer of (X,ω) under the $SL(2,\mathbb{R})$ action, i.e.

$$\Gamma(\omega) = \{g \in SL(2,\mathbb{R}) : g \cdot \omega = \omega\} < SL(2,\mathbb{R}).$$

$\Gamma(\omega)$ is always a non-cocompact discrete subgroup of $SL(2,\mathbb{R})$. Moreover, using the ergodicity of the $SL(2,\mathbb{R})$-action on $(\mathcal{H},\rho)$, one can show that for ρ-almost every (X,ω), the group $\Gamma(\omega)$ is trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. A surface (X,ω) is called a *Veech surface* if $\Gamma(\omega) < SL(2,\mathbb{R})$ is a lattice, i.e. the quotient space $SL(2,\mathbb{R})/\Gamma(\omega)$ has a finite $SL(2,\mathbb{R})$-invariant Radon measure. An equivalent characterization is the following result due to Smillie and Weiss [**SW10**].

Theorem 5.1

Let $(X,\omega) \in \mathcal{H}$. Then (X,ω) is a Veech surface if and only if the orbit $SL(2,\mathbb{R}) \cdot (X,\omega) \subseteq \mathcal{H}$ is closed.

The above discussion gives the identification

$$SL(2,\mathbb{R})/\Gamma(\omega) \simeq SL(2,\mathbb{R}) \cdot (X,\omega) = \overline{SL(2,\mathbb{R}) \cdot (X,\omega)}.$$

We now turn to counting. The first step is to slightly modify the Siegel–Veech transform. Given $f \in B_c(\mathbb{R}^2)$, using the identification $SL(2,\mathbb{R})/\Gamma(\omega) \simeq SL(2,\mathbb{R}) \cdot (X,\omega)$ established above, we may define

$$\widetilde{f} : SL(2,\mathbb{R})/\Gamma(\omega) \to \mathbb{C},\ g\Gamma(\omega) \mapsto \widehat{f}(g \cdot \omega).$$

Theorem 5.2

Let (X,ω) be a Veech surface and $\Gamma(\omega)$ its Veech group. Let μ be the normalized $SL(2,\mathbb{R})$-invariant Radon measure on $SL(2,\mathbb{R})/\Gamma(\omega)$. Then for any $f \in B_c(\mathbb{R}^2)$, there exists a constant $0 < c_{\Gamma(\omega)} < \infty$ such that

$$\int_{SL(2,\mathbb{R})/\Gamma(\omega)} \widetilde{f}(g\Gamma(\omega))\, d\mu = c_{\Gamma(\omega)} \int_{\mathbb{R}^2} f(x,y)\, dxdy.$$

Proof. The proof is identical to that of Theorem 2.1. □

The strategy of transforming the problem of counting saddle connections into one of studying certain ergodic averages, described in the previous sections, also applies here. In fact, it becomes simpler in the context of lattice surfaces, since we can replace the role played by Nevo's ergodic theorem with a classical result on the equidistribution of large circles in $SL(2,\mathbb{R})/\Gamma(\omega)$ (see Bekka and Mayer [**BM00**]).

Theorem 5.3

Let (X,ω) be a Veech surface. Then

$$\lim_{R\to+\infty} \frac{N(\omega,R)}{\pi R^2} = c_{\Gamma(\omega)}.$$

Remark 5.4 Historically, in [**V89**], Veech used ideas from Eisenstein series to address this problem, based on the following fact.

Theorem 5.5 ([**V89**])
Let (X, ω) be a Veech surface with Veech group $\Gamma(\omega)$. Then there exist vectors $v_1, \ldots, v_k \in \mathbb{R}^2 \setminus \{0\}$ such that

$$\Lambda_\omega = \bigsqcup_{i=1}^{k} \Gamma(\omega) \cdot v_i.$$

This theorem reduces the study of the asymptotics of $N(\omega, R)$ for the Veech surface (X, ω) to the asymptotics of

$$N_{\Gamma(\omega)}(v, R) = \# \left(\Gamma(\omega) \cdot v \cap B(0, R)\right).$$

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