From Limit theorem to Mixing limit theorem

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This is an extended version of the note accompanies the first part of the talk in the *Topics in zero entropy dynamics seminar* on the Mixing limit theorem. The main goal of my talk is to describe some methods for upgrading limit theorems for Birkhoff sums and cocycles over dynamical systems to mixing limit theorems following the work of Gouëzel [**G20**] and Arana-Herrera, Forni [**AF24**].

Contents

§0 Distributional limit theorem

Let $(X, \mathscr{B}, \mu, T)$ be an ergodic invertible probability measure-preserving system (p.m.p.s).

Birkhoff Sum

Let $f : X \to \mathbb{R}$ be a measurable function. The Birkhoff sums of f are said to satisfy a **(spatial) distributional limit theorem (DLT)** on $(X, \mathscr{B}, \mu)$ if there exist a sequence $\mathcal{A} = (A_N)_{N \in \mathbb{N}}$ of real numbers and a sequence $\mathcal{V} = (V_N)_{N \in \mathbb{N}}$ of positive real numbers with $V_N \to \infty$ as $N \to \infty$, and a random variable S on some probability space $(\Omega, \mathscr{F}, \mathbf{P})$, such that the random variables on $(X, \mathscr{B}, \mu)$:

$$S_N(x) := \frac{\sum_{n=0}^{N-1} f(T^n x) - A_N}{V_N}$$

converge in distribution to S as $N \to \infty$, that is, for every interval $(a, b) \subseteq \mathbb{R}$ with $\mathbf{P}(S \in \{a, b\}) = 0$, the following holds,

$$\lim_{N \to \infty} \mu\left(\{x \in X : \ S_N(x) \in (a, b)\}\right) = \mathbf{P}(S \in (a, b)).$$

We usually refer to $\mathcal{A}$ as the **averaging sequence**, $\mathcal{V}$ as the **normalizing sequence**, and S as the limiting distribution.

Example 0.1

Recall that the Birkhoff's ergodic theorem guarantees that for $f \in L^1(X, \mu)$

$$\lim_{N \to \infty} \mu\left(\left\{x \in X : \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int_X f \, d\mu\right\}\right) = 1.$$

Then the Birkhoff sums of f satisfy a DLT with averaging sequence $\mathcal{A} = (N \cdot \mu(f))$, normalizing sequence $\mathcal{V} = (N)$ and S a (constant) random variable with distribution the Dirac mass at $0 \in \mathbb{R}$.

Example 0.2 (Central limit theorem)

If the Birkhoff sums of f satisfy a DLT with $\mathcal{A} = (N \cdot \mu(f))$, $\mathcal{V} = (\sqrt{N})$ and S a mean 0 random variable with Gaussian distribution, then it is said to satisfies the central limit theorem. It characterize the distribution of the deviations of time averages from spatial averages.

Cocycle

We also care about the distributional limit theorems for cocycles. Let $C : X \times \mathbb{Z} \to \mathrm{GL}(m, \mathbb{R})$ be a measurable cocycle over T, satisfying

$$C(x, r+s) = C(T^r x, s) \cdot C(x, r), \quad x \in X, \ r, s \in \mathbb{Z}.$$

Denote by $\langle \cdot, \cdot \rangle$ the standard inner product on $\mathbb{R}^m$, by $\| \cdot \|$ the corresponding Euclidean norm, and by η the induced probability measure on the projectivization $\mathbb{P}\mathbb{R}^m$. The projectivized bundle $X \times \mathbb{P}\mathbb{R}^m$ is endowed with the product measure $\mu \otimes \eta$.

Given $(x,v) \in X \times (\mathbb{R}^m \setminus \{0\})$ or $X \times \mathbb{PR}^m$ and $N \in \mathbb{N}$, consider the quantities

$$\sigma(x,v,N) := \log \frac{\|C(x,N)v\|}{\|v\|} \in \mathbb{R},$$
$$\sigma(x,N) := \log \sup_{v \in \mathbb{R}^m \setminus \{0\}} \frac{\|C(x,N)v\|}{\|v\|} \in \mathbb{R}.$$

We say the cocycle C is log-integrable if the following maps belong to $L^1(X,\mu)$,

$$x \mapsto \max\{0, \sigma(x,1)\}, \quad x \mapsto \max\{0, \sigma(x,-1)\}.$$

We say the cocycle C satisfies a (spatial) distributional limit theorem (DLT) on $(X \times \mathbb{PR}^m, \mu \otimes \eta)$ if there exist sequences $\mathcal{A} := (A_N)_{N\in\mathbb{N}}$ of real numbers and $\mathcal{V} := (V_N)_{N\in\mathbb{N}}$ of positive real numbers with $V_N \to \infty$ as $N \to \infty$, and a random variable S, such that the random variables

$$S_N(x,v) = \frac{\sigma(x,v,N) - A_N}{V_N} \quad \text{on } (X \times \mathbb{PR}^m, \mu \otimes \eta)$$

converge in distribution to S as $N \to \infty$, i.e., for every interval $(a,b) \subseteq \mathbb{R}$ such that $\mathbf{P}(S \in \{a,b\}) = 0$, the following holds,

$$\lim_{N\to\infty} (\mu \otimes \eta)\big(\{(x,v) \in X \times \mathbb{PR}^m \mid S_N(x,v) \in (a,b)\}\big) = \mathbf{P}(S \in (a,b)).$$

We usually refer to $\mathcal{A}$ as the averaging sequence, $\mathcal{V}$ as the normalizing sequence, and S as the limiting distribution.

Example 0.3

The Oseledets ergodic theorem guarantees that, every log-integrable cocycle $C : X \times \mathbb{Z} \to \mathrm{GL}(m,\mathbb{R})$ with top Lyapunov exponent $\lambda \in \mathbb{R}$ satisfies a DLT with averaging sequence $\mathcal{A} = (N \cdot \lambda)$, normalizing sequence $\mathcal{V} = (N)$, and S a (constant) random variable with distribution equal to the Dirac mass at $0 \in \mathbb{R}$.

Some Motivations and Goals

When given a system where some observable/ cocycle satisfies a certain distributional limit theorem (DLT), we are naturally led to consider its statistical 'robustness' under different observational conditions. In physical situations, where there is an a priori given reference probability measure ν (for instance, Lebesgue measure) which perhaps differs from the invariant measure μ, there can be a discussion of whether it is more natural to consider such a distributional convergence with respect to the reference measure ν or to the invariant measure μ. This directly motivates our first question:

Question 1: If the DLT is established under the invariant measure μ, can we deduce the same asymptotic distributional information of the system under the physical measure ν?

Also, one may be curious about what if there is conditioning on the initial state/ final state or even multiple states along the orbit. This leads to our second question:

Question 2: Does the limiting distribution remain unchanged if we prescribe the initial and final positions of points?

The goal of this note is to address these two questions within specific dynamical contexts. It should be noted that we do not discuss the techniques required to establish a distributional limit theorem itself; for an introduction of how to prove such results, the interested reader may refer to, for instance, [P22].

§1 Birkhoff Sum

We answer the two questions for the case of birkhoff sum in this section.

Regarding Question 1, we show that if the reference measure ν is absolutely continuous with respect to the invariant measure μ (denoted by $\nu \ll \mu$), the system retains the exact same asymptotic distributional profile under the physical measure ν. This result is a consequence of a classical theorem by Eagleson [E77], for which we provide a self-contained proof.

As for Question 2, while conditioning solely on the initial (or final) position follows as a direct corollary of Eagleson's theorem, the case of simultaneous conditioning on both initial and final positions is more subtle. We demonstrate that the limiting distribution remains invariant under such two-point constraints provided the system is further assumed to be mixing. This result follows the framework established in [G20].

Eagleson's Theorem

Theorem 1.1 (Eagleson's Theorem)
Let $(X, \mathscr{B}, \mu, T)$ be an ergodic p.m.p.s, and $f : X \to \mathbb{R}$ be a measurable function. Suppose the ergodic sums of f satisfy a DLT on $(X, \mathscr{B}, \mu)$ with averaging sequence $\mathcal{A} = (A_N)$, normalizing sequence $\mathcal{V} = (V_N)$ and limiting distribution S. Then for any probability measure ν on $(X, \mathscr{B})$ that is **absolutely continuous** with μ, the ergodic sums of f as a random variable on $(X, \mathscr{B}, \nu)$ satisfy the same DLT as on $(X, \mathscr{B}, \mu)$. That is, for every interval $(a, b) \subseteq \mathbb{R}$ with $\mathbf{P}(S \in \{a, b\}) = 0$, the following holds,

$$\lim_{N\to\infty} \nu\left(\{x \in X : S_N(x) \in (a, b)\}\right) = \mathbf{P}(S \in (a, b)). \tag{1}$$

Remark 1.2 Let $U \in \mathscr{B}$ with $\mu(U) > 0$. If we take $\nu = \frac{1}{\mu(U)}\mu(\cdot|U)$, then in this case, (1) means

$$\lim_{N\to\infty} \mu\left(\{x \in X : S_N(x) \in (a, b),\ x \in U\}\right) = \mu(U)\mathbf{P}(S \in (a, b)).$$

It tells us we can prescribe the initial position of points without changing the limiting distribution (up to a scalar).

We can also conditioning at time n by applying Eagleson's theorem in the natural extension and a change of variables.

To prove the theorem, we will express the distributional convergence through the following classical lemma.

Lemma 1.3
A sequence of real random variables Z_n (**not necessarily coming from the same probability space!**) converges in distribution to Z if and only if, for any function $\alpha : \mathbb{R} \to \mathbb{R}$ which is bounded and Lipschitz, $\mathbf{E}[\alpha(Z_n)] \to \mathbf{E}[\alpha(Z)]$.

We will also use the following simple but important observation in the proof, which relies on the additive property of the Birkhoff sum and the fact T preserves μ.

Lemma 1.4

Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz function and let $\rho \in L^\infty(X, \mu)$. For any fixed integer $k \in \mathbb{N}$ and any $1 \le j \le k$, we have

$$\int_X \alpha(S_N)\rho \circ T^j \, d\mu = \int_X \alpha(S_N)\rho \, d\mu + o_{\alpha,\rho,k}(1), \quad \text{as } N \to \infty$$

Proof. Fix $k \in \mathbb{N}$. For any $1 \le j \le k$, the T-invariance of μ implies that

$$\int_X \alpha(S_N)\rho \, d\mu = \int_X \alpha(S_N \circ T^j)(\rho \circ T^j) \, d\mu.$$

Then for $N > k$, we can bound the difference as follows:

$$\begin{aligned}\left| \int_X \alpha(S_N)\rho \circ T^j \, d\mu - \int_X \alpha(S_N)\rho \, d\mu \right| &= \left| \int_X \Big(\alpha(S_N) - \alpha(S_N \circ T^j)\Big)\rho \circ T^j \, d\mu \right| \\ &\le \int_X \left|\alpha(S_N) - \alpha(S_N \circ T^j)\right| \cdot \left|\rho \circ T^j\right| \, d\mu \\ &\le \mathrm{Lip}(\alpha) \, \|\rho\|_{L^\infty} \int_X \left| \frac{\sum_{n=0}^{N-1} f \circ T^{n+j} - \sum_{n=0}^{N-1} f \circ T^n}{V_N} \right| d\mu \\ &\le \frac{2j\mathrm{Lip}(\alpha) \, \|\rho\|_{L^\infty} \, \|f\|_{L^1}}{V_N}.\end{aligned}$$

To justify the final inequality, we observe that the difference of the Birkhoff sums telescopes:

$$\left| \sum_{n=0}^{N-1} f \circ T^{n+j} - \sum_{n=0}^{N-1} f \circ T^n \right| = \left| \sum_{l=0}^{j-1} \Big(f \circ T^{N+l} - f \circ T^l \Big) \right| \le \sum_{l=0}^{j-1} \Big(\left| f \circ T^{N+l} \right| + \left| f \circ T^l \right| \Big).$$

Integrating this pointwise bound and applying the T-invariance of μ yields a constant $2j\|f\|_{L^1}$. Since $V_N \to \infty$ as $N \to \infty$, the bounded term divided by V_N vanishes, which concludes the proof:

$$\left| \int_X \alpha(S_N)\rho \circ T^j \, d\mu - \int_X \alpha(S_N)\rho \, d\mu \right| = o_{\alpha,\rho,k}(1).$$

□

We are now ready to prove Theorem 1.1. The key idea behind the proof echoes a central philosophy we learned from the ergodic theorem: while an observable might exhibit erratic behavior at any specific instant, its long-term time average becomes well-behaved.

Proof of Theorem 1.1. Fix a bounded Lipschitz function $\alpha : \mathbb{R} \to \mathbb{R}$. Our goal is to show that

$$\mathbf{E}_\nu[\alpha(S_N)] = \int_X \alpha(S_N) \, d\nu \xrightarrow{N \to \infty} \mathbf{E}[\alpha(S)].$$

Since $\nu \ll \mu$, let $\rho = \frac{d\nu}{d\mu} \in L^1(X, \mu)$ be the Radon-Nikodym derivative, and define $\rho_0 = \rho - 1 \in L^1_0(X, \mu)$. Then,

$$\mathbf{E}_\nu[\alpha(S_N)] = \int_X \alpha(S_N) \, d\mu + \int_X \alpha(S_N)\rho_0 \, d\mu.$$

By our initial assumption, the first term on the right-hand side converges to $\mathbf{E}[\alpha(S)]$. Hence, it remains to show that the second term converges to 0 as $N \to \infty$.

Fix $\varepsilon > 0$. Since $L_0^\infty(X,\mu)$ is dense in $L_0^1(X,\mu)$, we can choose a function $\tilde{\rho}_0 \in L_0^\infty(X,\mu)$ such that

$$\|\rho_0 - \tilde{\rho}_0\|_{L^1} < \varepsilon.$$

Since $\int_X \tilde{\rho}_0 \, d\mu = 0$ and the system is ergodic, we can pick an integer $k \in \mathbb{N}$ large enough so that the von Neumann Mean Ergodic Theorem yields:

$$\left\| \frac{1}{k} \sum_{j=0}^{k-1} \tilde{\rho}_0 \circ T^j \right\|_{L^2} \leq \varepsilon.$$

We now split the integral into two parts and apply Lemma 1.4:

$$\begin{aligned} \left| \int_X \alpha(S_N)\rho_0 \, d\mu \right| &\leq \left| \int_X \alpha(S_N)(\rho_0 - \tilde{\rho}_0) \, d\mu \right| + \left| \int_X \alpha(S_N)\tilde{\rho}_0 \, d\mu \right| \\ &\leq \|\alpha\|_{L^\infty} \|\rho_0 - \tilde{\rho}_0\|_{L^1} + \left| \int_X \alpha(S_N) \left(\frac{1}{k} \sum_{j=0}^{k-1} \tilde{\rho}_0 \circ T^j \right) d\mu \right| + o_{\alpha,\tilde{\rho}_0,k}(1). \end{aligned}$$

For the middle term, we apply the Cauchy-Schwarz inequality:

$$\begin{aligned} \left| \int_X \alpha(S_N)\rho_0 \, d\mu \right| &\leq \varepsilon \|\alpha\|_{L^\infty} + \|\alpha\|_{L^2} \left\| \frac{1}{k} \sum_{j=0}^{k-1} \tilde{\rho}_0 \circ T^j \right\|_{L^2} + o_{\alpha,\tilde{\rho}_0,k}(1) \\ &\leq \varepsilon \|\alpha\|_{L^\infty} + \varepsilon \|\alpha\|_{L^\infty} + o_{\alpha,\tilde{\rho}_0,k}(1) \\ &= 2\varepsilon \|\alpha\|_{L^\infty} + o_{\alpha,\tilde{\rho}_0,k}(1). \end{aligned}$$

Taking the limit supremum as $N \to \infty$, the $o(1)$ term vanishes, leaving:

$$\limsup_{N\to\infty} \left| \int_X \alpha(S_N)\rho_0 \, d\mu \right| \leq 2\varepsilon \|\alpha\|_{L^\infty}.$$

Since $\varepsilon > 0$ was chosen arbitrarily, the integral converges to 0. This completes the proof. □

Simultaneous conditioning: mixing limit theorem

Theorem 1.5 (Mixing limit theorem for Birkhoff sum, [G20; AF24])
Let $(X, \mathscr{B}, \mu, T)$ be an **mixing** p.m.p.s, and $f : X \to \mathbb{R}$ be a measurable function. Suppose the Birkoff sums of f satisfy a DLT on $(X, \mathscr{B}, \mu)$ with averaging sequence $\mathcal{A} = (A_N)$, normalizing sequence $\mathcal{V} = (V_N)$ and limiting distribution S. Let $\varphi_1, \varphi_2 : X \to \mathbb{R}$ be two non-negative L^∞ functions with $\int \varphi_i d\mu = 1$. Define a sequence of measures ν_N by

$$\nu_N(U) = \int_U \varphi_1 \cdot \varphi_2 \circ T^N d\mu, \ \forall U \in \mathscr{B}.$$

Then the random variables S_N from the probability spaces $(X, \rho_N = \frac{1}{\nu_N(X)}\nu_n)$ converges in distribution to S.

Remark 1.6 If we let $U_1, U_2 \in \mathscr{B}$ with $\mu(U_1), \mu(U_2) > 0$ and let $\varphi_i = \frac{1}{\mu(U_i)}\chi_{U_i}$. The distributional convergence in theorem 1.5 give us for every interval $(a, b) \subseteq \mathbb{R}$ with

$\mathbf{P}(S \in \{a,b\}) = 0$,

$$\lim_{N\to\infty} \frac{\mu\left(\{x \in X:\ S_N(x) \in (a,b),\ x \in U_1,\ T^N x \in U_2\}\right)}{\mu(U_1 \cap T^{-N}U_2)} = \mathbf{P}(S \in (a,b)).$$

Since the system is mixing, $\mu(U_1 \cap T^{-N}U_2) = \mu(U_1)\mu(U_2) + o(1)$ as $N \to \infty$. Therefore,

$$\lim_{N\to\infty} \mu\left(\left\{x \in X:\ S_N(x) \in (a,b),\ x \in U_1,\ T^N x \in U_2\right\}\right) = \mu(U_1)\mathbf{P}(S \in (a,b))\mu(U_2).$$

To prove the theorem, by lemma 1.3, it is enough to show that

$$\int_X \alpha(S_N) \cdot (\varphi_1 \cdot \varphi_2 \circ T^N)\, d\mu \overset{N\to\infty}{\longrightarrow} \left(\int_X \varphi_1\, d\mu\right) \mathbf{E}(\alpha(S)) \left(\int_X \varphi_1\, d\mu\right) \tag{2}$$

for any $\alpha : \mathbb{R} \to \mathbb{R}$ which is bounded and Lipschitz. Note that when φ_2 is constant, the convergence (2) holds by Eagleson's theorem 1.1. Hence, we can without loss of generality replace φ_2 with $\varphi_2 - \int \varphi_2 d\mu$ and assume $\int \varphi_2 d\mu = 0$. Our goal then becomes

$$\int_X \alpha(S_N) \cdot (\varphi_1 \cdot \varphi_2 \circ T^N)\, d\mu \overset{N\to\infty}{\longrightarrow} 0$$

Note that in the proof of Lemma 1.4, the constant only depends on $\|\rho\|_{L^\infty}$ instead of ρ and $\|\varphi_1 \cdot \varphi_2 \circ T^N\|_{L^\infty} \le \|\varphi_1\|_{L^\infty}\|\varphi_2\|_{L^\infty}$. Hence for any $k \in \mathbb{N}$, we still have

$$\int_X \alpha(S_N)\left(\frac{1}{k}\sum_{j=0}^{k-1} \varphi_1 \circ T^j \cdot \varphi_2 \circ T^{N+j}\right) = \int_X \alpha(S_N) \cdot (\varphi_1 \cdot \varphi_2 \circ T^N)\, d\mu + o_{\alpha,\varphi_1,\varphi_2,k}(1).$$

Then the following lemma will help us complete the proof.

Lemma 1.7

Let $(X, \mathscr{B}, \mu, T)$ be a mixing p.m.p.s and $\psi_1, \psi_2 \in L^\infty(X,\mu)$ with $\int_X \psi_2\, d\mu = 0$. Then

$$\limsup_{N\to\infty} \left\| \frac{1}{k}\sum_{j=0}^{k-1} \psi_1 \circ T^j \cdot \psi_2 \circ T^{N+j} \right\|_{L^2} = o(1), \quad \text{as } k \to \infty$$

Proof. Let $A_{k,N} = \frac{1}{k}\sum_{j=0}^{k-1} \psi_1 \circ T^j \cdot \psi_2 \circ T^{N+j}$. Expanding the square, we have

$$\begin{aligned}\|A_{k,N}\|_{L^2}^2 = &\frac{1}{k^2}\sum_{j=0}^{k-1}\int_X (\psi_1 \circ T^j)^2 (\psi_2 \circ T^{N+j})^2\, d\mu \\ &+ \frac{2}{k^2}\sum_{j=1}^{k-1}\sum_{i<j} (\psi_1 \circ T^i \cdot \psi_1 \circ T^j)(\psi_2 \circ T^{N+i} \cdot \psi_2 \circ T^{N+j}) d\mu.\end{aligned}$$

By the T-invariance of μ, the first term (diagonal sum) is bounded by $\frac{1}{k}\|\psi_1\|_{L^\infty}^2\|\psi_2\|_{L^\infty}^2$.

For the second term, also by the T-invariance of μ,

$$\begin{aligned}&\frac{2}{k^2}\sum_{j=1}^{k-1}\sum_{i<j} (\psi_1 \circ T^i \cdot \psi_1 \circ T^j)(\psi_2 \circ T^{N+i} \cdot \psi_2 \circ T^{N+j}) d\mu = \\ &\quad + \frac{2}{k}\sum_{j=1}^{k-1}\left(1 - \frac{j}{k}\right)\int_X (\psi_1 \cdot \psi_1 \circ T^j) \cdot ((\psi_2 \cdot \psi_2 \circ T^j) \circ T^N)\, d\mu.\end{aligned}$$

Fix $k \in \mathbb{N}$. Since T is mixing, as $N \to \infty$, the integral in the finite sum asymptotically decouples:

$$\int_X (\psi_1 \cdot \psi_1 \circ T^j) \cdot ((\psi_2 \cdot \psi_2 \circ T^j) \circ T^N)\, d\mu = \left(\int_X \psi_1 \cdot \psi_1 \circ T^j\, d\mu\right)\left(\int_X \psi_2 \cdot \psi_2 \circ T^j\, d\mu\right) + o(1).$$

Taking the limit supremum as $N \to \infty$, the finite number of $o(1)$ terms vanish. Bounding the first integral by $\|\psi_1\|_{L^\infty}^2$, we obtain:

$$\limsup_{N\to\infty} \|A_{k,N}\|_{L^2}^2 \le \frac{\|\psi_1\|_{L^\infty}^2 \|\psi_2\|_{L^\infty}^2}{k} + \frac{2\|\psi_1\|_{L^\infty}^2}{k} \sum_{j=1}^{k-1} \left| \int_X \psi_2 \cdot \psi_2 \circ T^j\, d\mu \right|.$$

Let $c_j = \left|\int_X \psi_2 \cdot \psi_2 \circ T^j\, d\mu\right|$. Because T is mixing and $\int_X \psi_2\, d\mu = 0$, we know that $c_j = o(1)$ as $j \to \infty$. Then the Cesàro mean c_j also converges to 0. Therefore,

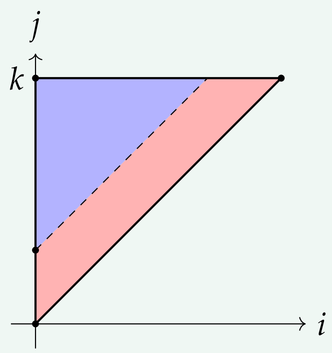
$$\frac{1}{k}\sum_{j=1}^{k-1} c_j = o(1) \quad \text{as } k \to \infty.$$

Combining these estimates, we conclude:

$$\limsup_{N\to\infty} \|A_{k,N}\|_{L^2}^2 \le \frac{C}{k} + 2\|\psi_1\|_{L^\infty}^2 \cdot o(1) = o(1), \quad \text{as } k \to \infty$$

where $C = \|\psi_1\|_{L^\infty}^2 \|\psi_2\|_{L^\infty}^2$. □

Remark 1.8 The proof of the above lemma can be visualized as follows:



The terms falling within the blue region are controlled by the mixing property. Conversely, the terms in the red region (the diagonal and short-range terms) are suppressed by taking k to be sufficiently large, as their overall contribution becomes a vanishingly small fraction of the total area.

§2 Cocycles

In this section we extend the framework from scalar observables to matrix-valued cocycles. Let $C : X \times \mathbb{Z} \to \mathrm{GL}(m, \mathbb{R})$ be a measurable cocycle over T. We are interested in the following quantities: given $(x, v) \in X \times (\mathbb{R}^m \setminus \{0\})$ or $X \times \mathbb{PR}^m$ and $N \in \mathbb{N}$,

$$\sigma(x, v, N) := \log \frac{\|C(x,N)v\|}{\|v\|} \in \mathbb{R},$$

$$\sigma(x, N) := \log \sup_{v \in \mathbb{R}^m \setminus \{0\}} \frac{\|C(x,N)v\|}{\|v\|} \in \mathbb{R}.$$

Recall that the cocycle C is log-integrable if the following functions belong to $L^1(X,\mu)$,

$$x \mapsto \max\{0, \sigma(x,1)\}, \quad x \mapsto \max\{0, \sigma(x,-1)\}.$$

Let η be the induced probability measure on the projectivization $\mathbb{PR}^m$. For any $\Psi \in L^1(X \times \mathbb{PR}^m, \mu \otimes \eta)$, we introduce

$$\mathbb{A}\Psi : X \to X,\ x \longmapsto \int_{\mathbb{PR}^m} \Psi(x,v)\, d\eta(v).$$

Consider the normalized random variable $S_N(x,v) = \frac{\sigma(x,v,N)-A_N}{V_N}$ on $(X \times \mathbb{PR}^m, \mu \otimes \eta)$. The goal of this section is to show that under some mild ergodic condition on the cocycles, the two theorems we established for Birkhoff sums are also true in the case of cocycles.

Definition 2.1 — Let $\mathcal{V} = (V_N) \in \mathbb{R}_+^{\mathbb{N}}$. We say the cocycle C has $\mathcal{V}$-simple-dominated-splitting if for μ-almost every $x \in X$ and η-almost every $v, w \in \mathbb{PR}^m$, the following holds,

$$|\sigma(x,v,N) - \sigma(x,w,N)| = o_{x,v,w}(V_N), \quad \text{as } N \to \infty.$$

Example 2.2

The log-integrable cocycle $C : X \times \mathbb{Z} \to \mathrm{GL}(m,\mathbb{R})$ over $(X, \mathscr{B}, \mu, T)$ with **simple top Lyapunov exponent** $\lambda \in \mathbb{R}$ has simple-dominated-splitting with respect to any diverging sequence $\mathcal{V}$.

The purpose of introducing the definition of $\mathcal{V}$-simple-dominated-splitting is to eliminate the effect of the choice of the direction for almost every base point so that we can deduce the similar result as Lemma 1.4.

Lemma 2.3

Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz function and let $\rho \in L^\infty(X,\mu)$. For any fixed integer $k \in \mathbb{N}$ and any $1 \le j \le k$, we have

$$\int_X \mathbb{A}(\alpha(S_N))\rho \circ T^j\, d\mu = \int_X \mathbb{A}(\alpha(S_N))\rho\, d\mu + o_{C,\alpha,\rho,k}(1), \quad \text{as } N \to \infty.$$

Sketch of proof. Fix $k \in \mathbb{N}$. For any $1 \le j \le k$, the T-invariance of μ implies that

$$\int_X \mathbb{A}(\alpha(S_N))\rho\, d\mu = \int_X \mathbb{A}(\alpha(S_N)) \circ T^j (\rho \circ T^j)\, d\mu.$$

Then for $N > k$, as in the proof of Lemma 1.4, the key is to understand the following integral

$$\int_X \mathbb{A}(\alpha(S_N)) \circ T^j - \mathbb{A}(\alpha(S_N))\, d\mu.$$

By definition,

$$\begin{aligned}
&\left|\int_X \mathbb{A}(\alpha(S_N))\circ T^j - \mathbb{A}(\alpha(S_N))\, d\mu\right| \\
=&\left|\int_X \int_{\mathbb{PR}^m} \alpha(S_N(T^j x, v)) - \alpha(S_N(x,v)) d\eta(v) d\mu(x)\right| \\
\leq &\mathrm{Lip}(\alpha) \int_X \int_{\mathbb{PR}^m} \frac{|\sigma(T^j x, v, N) - \sigma(x,v,N)|}{V_N}\, d\eta(v) d\mu(x)
\end{aligned}$$

By the cocycle identity, $C(T^j x, N) = C(x, N+j)C(x,j)^{-1}$. Let $w_{x,j} = \frac{C(x,j)^{-1}v}{\|C(x,j)^{-1}v\|} \in \mathbb{PR}^m$, then

$$\sigma(T^j x, v, N) = \log \frac{\|C(T^j x, N)v\|}{\|v\|} = \log \frac{\|C(x, N+j)w_{x,j}\|}{\|w_{x,j}\|} + \log \frac{\|w_{x,j}\|}{\|v\|}.$$

Therefore

$$\begin{aligned}
|\sigma(T^j x, v, N) - \sigma(x,v,N)| &= \left|\sigma(x, w_{x,j}, N+j) - \sigma(x,v,N) + \log \frac{\|w_{x,j}\|}{\|v\|}\right| \\
&\leq |\sigma(x, w_{x,j}, N+j) - \sigma(x,v,N+j)| \\
&\quad + |\sigma(x,v,N+j) - \sigma(x,v,N)| + \left|\log \frac{\|w_{x,j}\|}{\|v\|}\right|
\end{aligned}$$

We integrate the right hand side and divide it by V_N and take the limit as $N \to \infty$:

- For the first term, the $\mathcal{V}$-simple-dominated-splitting property guarantees that for μ-a.e. x and η-a.e. v, the difference is $o(V_{N+j})$. Since $V_{N+j} \sim V_N$, this term is of $o(V_N)$. After dividing V_N, it is of $o(1)$.
- For the second term, $\sigma(x,v,N+j) - \sigma(x,v,N) = \sigma(T^N x, C(x,N)v, j)$. This is bounded by $\sigma(T^N x, j)$. Since C is log-integrable, this is an L^1 function. As $V_N \to \infty$, an L^1 observable divided by V_N converges to 0 almost surely.
- The third term $\log \|C(x,j)^{-1}v\|$ is independent of N, so dividing by V_N obviously sends it to 0.

□

Once we had this lemma, using the same argument as in the proof of Theorem 1.1, we can have:

Theorem 2.4 (Eagleson's Theorem for Cocycles,[**AF24**])
Let $(X, \mathscr{B}, \mu, T)$ be an ergodic p.m.p.s, and $C : X \times \mathbb{Z} \to \mathrm{GL}(m, \mathbb{R})$ a log-integrable cocycle with $\mathcal{V}$-simple-dominated-splitting. Suppose C satisfies a DLT on $(X \times \mathbb{PR}^m, \mu \otimes \eta)$ with averaging sequence $\mathcal{A} = (A_N)$, normalizing sequence $\mathcal{V} = (V_N)$ and limiting distribution S. Then for any probability measure $\nu \ll \mu$ on the base space X, the DLT holds on $(X \times \mathbb{PR}^m, \nu \otimes \eta)$ with the same limit S.

Similarly, under the assumption of the base system is mixing, we can have

Theorem 2.5 (Mixing limit theorem for Cocycles,[**AF24**])
Let $(X, \mathscr{B}, \mu, T)$ be an **mixing** p.m.p.s, and $C : X \times \mathbb{Z} \to \mathrm{GL}(m, \mathbb{R})$ be a a log-integrable cocycle with $\mathcal{V}$-simple-dominated-splitting. Suppose C satisfies a DLT on $(X \times \mathbb{PR}^m, \mu \otimes \eta)$ with averaging sequence $\mathcal{A} = (A_N)$, normalizing sequence $\mathcal{V} = (V_N)$ and limiting distribution S. Let $\varphi_1,\ \varphi_2 : X \to \mathbb{R}$ be two non-negative L^∞ functions with $\int \varphi_i d\mu = 1$. Define a sequence of measures ν_N by

$$\nu_N(U) = \int_U \varphi_1 \cdot \varphi_2 \circ T^N d\mu,\ \forall U \in \mathscr{B}.$$

Then the random variables S_N from the probability spaces $(X \times \mathbb{PR}^m, \rho_N = \frac{1}{\nu_N(X)} \nu_n \otimes \eta)$ converges in distribution to S

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