The Geometric Multiplicative Ergodic Theorem

Jianshu Jiang

November 15, 2025

These are an extended version of the notes accompanies the third talk in the *Topics in zero entropy dynamics seminar* on the Oseledets' Multiplicative Ergodic Theorem (MET). We try to discribe a geometric framework of this theorem pioneered by Kaimanovich. We show that the MET is equivalent to a statement about the asymptotic behavior of a 'random walk' on the symmetric space $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R})$, which would be identified with the space of symmetric positive definite matrices with determinant 1, specifically, that the random walk being "sublinearly tracked" by an ideal geodesic ray. We mainly follow the notes on related topics by S. Filip and A. Karlsson

§1 Warn up: Geometric MET in dimension 2

Let us begin with a completely geometric setting. Let $(\Omega, \mathcal{B}, \mu, T)$ be an ergodic probability measure preserving system, and $A : \Omega \to \mathrm{SL}_2(\mathbb{R})$ a measurable linear cocycle over T satisfying the integrability condition

$$\int_\Omega \log^+ \|A(\omega)\|_{op}\, d\mu(\omega) < \infty. \tag{1}$$

where the operator norm of $g \in \mathrm{GL}_d(\mathbb{R})$ is defined as

$$\|g\|_{op} := \sup_{v \in \mathbb{R}^d \setminus \{0\}} \frac{\|gv\|}{\|v\|} = \sqrt{\text{the maximal eigenvalue of } gg^t}.$$

For each $\omega \in \Omega$, let

$$A_\omega^{(n)} := A(\omega)A(T\omega)\cdots A(T^{n-1}\omega), \quad n \geq 1.$$

Recall that $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$, equipped with hyperbolic metric

$$ds = \frac{|dz|}{y}, \quad \text{where } z = x + iy.$$

isometrically by Möbius transformations:

$$g \cdot z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

We also have the *Poincaré disk model*: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the hyperbolic metric

$$ds = \frac{2|dz|}{1 - |z|^2}.$$

These two models are isometric through the map

$$\phi : \mathbb{H} \to \mathbb{D},\ \phi(z) = \frac{z-i}{z+i}.$$

Via the isometry ϕ, $\mathrm{SL}_2(\mathbb{R})$ also acts by isometries on the disk $\mathbb{D}$. Therefore,

$$\mathrm{SL}_2(\mathbb{R}) \simeq \mathrm{Isom}(\mathbb{H}) \simeq \mathrm{Isom}(\mathbb{D}).$$

Write o to be the center of the disk $\mathbb{D}$. Each point $\omega \in \Omega$ gives us a sequence $\{A_\omega^{(n)} \cdot o\}_{n\in\mathbb{N}}$ in $\mathbb{D}$ via the cocycle A. We are interested in how do these sequences generically look like?

Theorem 1.1

Write $x_n(\omega) = A_\omega^{(n)} \cdot o,\ n \geq 1$. Suppose

$$\int_\Omega d(A(\omega)\cdot o, o)\, d\mu(\omega) < +\infty,$$

then there exists $\Omega' \subseteq \Omega$ with $\mu(\Omega') = 1$ and $L \geq 0$ such that for every $\omega \in \Omega'$, we have

- **Small steps**: $d(x_n(\omega), x_{n+1}(\omega)) = o(n)$,
- **Linear growth of distance** : $d(x_n(\omega), o) = Ln + o(n)$.

Moreover, if $L > 0$, then for every $\omega \in \Omega'$, the sequence $\{x_n(\omega)\}_{n\in\mathbb{N}}$ is *sublinearly tracked* by a unique ideal geodesic ray, that is, there exists a unique ideal geodesic ray $\gamma_\omega : [0, \infty) \to \mathbb{D}$ (i.e. $d(\gamma_\omega(0), \gamma_\omega(t)) = t$) starting from o such that

$$d(x_n(\omega), \gamma_\omega(L\cdot n)) = o(n).$$

Proof. For $n \geq 1$, let $f_n(\omega) = d(x_n(\omega), o)$. Notice that $f_1 = d(x_1(\cdot), o) = d(A(\cdot) \cdot o, o) \in L^1(\Omega, \mu)$. Since $\mathrm{SL}_2(\mathbb{R})$ acts by isometry, $d(x_n(\omega), x_{n+1}(\omega)) = d(o, A(\omega) \cdot o)$. By Birkhoff ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n}(d(x_n(\omega), x_{n+1}(\omega)) = \lim_{n\to\infty} \frac{1}{n} d(o, A(\omega)\cdot o) = 0, \ a.e. \omega \in \Omega$$

Hence the first statement holds.

For the second statement, again using isometric action and triangle inequality we have

$$\begin{aligned} f_{m+n}(\omega) = d(x_{m+n}(\omega), o) &\leq d(x_{m+n}(\omega), x_n(\omega)) + d(x_n(\omega), o) \\ &= d(x_m(T^n\omega), o) + d(x_n(\omega), o) = f_m(T^n\omega) + f_n(\omega). \end{aligned}$$

Then by Kingman's subadditive ergodic theorem, there exists some $L \geq 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} d(x_n(\omega), o) = L, \ a.e. \omega \in \Omega$$

Finally, suppose $L > 0$, recall the Law of Sines in constant negative curvature. Consider a triangle with angles of sizes α, β, γ and opposite sides of lengths a, b, c. These quantities are related by

$$\frac{\sin \alpha}{\sinh a} = \frac{\sin \beta}{\sinh b} = \frac{\sin \gamma}{\sinh c}.$$

Recall that we have the estimates

- $d(x_n(\omega), x_{n+1}(\omega)) = o(n)$,
- $d(x_n(\omega), o) = Ln + o(n)$.

Define the angle, when viewed from the origin, between successive points $\alpha_n(\omega) = \angle x_n o x_{n+1}$, also define the angle $\beta_n = \angle o x_n x_{n+1}$. The law of sines and previous estimates then give

$$\sin \alpha_n = \sin \beta_n \cdot \frac{\sinh d(x_n, x_{n+1})}{\sinh d(x_{n+1}, o)} \leq \frac{\sinh d(x_n, x_{n+1})}{\sinh d(x_{n+1}, o)} = e^{-Ln + o(n)}.$$

This implies $\alpha := \sum_{n\geq 1} \alpha_n$ converges uniformly. Therefore, the sequence of angles $\{\tilde{\alpha}_n = \angle x_0 o x_n\}$ is a Cauchy sequence. Let $\gamma_\omega : [0, \infty) \to \mathbb{D}$ be the ideal geodesic ray starting from o with direction $\tilde{\alpha} := \lim_{n\to\infty} \tilde{\alpha}_n$ would be what we want. $\square$

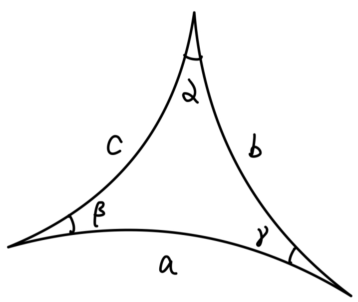

Figure 1: Law of Sines in constant negative curvature

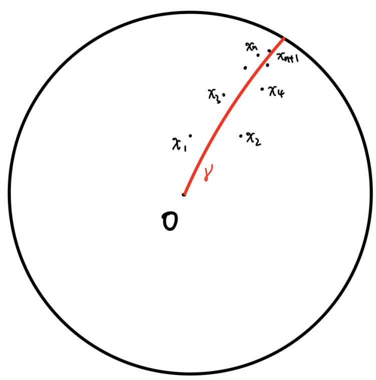

Figure 2: Sublinear track by a geodesic ray

Remark 1.2 (i). Given $z \in \mathbb{D}$, one can explicitly write down the formula of its distance to o as

$$d(z, o) = \log \frac{1 + |z|}{1 - |z|}.$$

Using this (plus some computation), one can show that

$$d(A(\omega) \cdot o, o) \asymp \log \|A(\omega)\|_{op}.$$

As a result, the integrability condition in the theorem is equivalent to (1).

(ii). Sometimes, the sequence being sublinearly tracked by a unique ideal geodesic ray will also be called converging to the boundary, which we will discuss here. But it turns out that the direction of the geodesic ray (or the limit on the boundary) will encode the filtration in Oseledets theorem.

We end this section by giving the following definition.

Definition 1.3 — Let (X, d) be a complete geodesic metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called a *regular sequence* if there exists a geodesic ray $\gamma : [0, \infty) \to X$ and $\theta \geq 0$ such that

$$d(x_n, \gamma(\theta \cdot n)) = o(n).$$

§2 Dictionary for symmetric spaces

The aim of this section is to build the necessary language and introduce some basic results about symmetric spaces, leading to stating and proving our main results: Kaimanovich's criteria on regularity in symmetric spaces. We will careful treat the most important example for us, namely $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R})$ which will be identified with the space of symmetric positive definite matrices with determinant 1. For a survey of symmetric space from a more geometric point of view, we refer readers to G. Link's survey *An introduction to globally symmetric spaces*. For a detailed introduction, we refer readers to the well written notes of A. Iozzi.

Structure theory of Lie groups

Let $G < \mathrm{GL}_d(\mathbb{R})$ be a semisimple, closed subgroup that is stable under transposition and not contained in $\mathrm{O}_d(\mathbb{R})$. Define the involutive automorphism

$$\sigma : G \to G, \quad g \mapsto (g^t)^{-1}.$$

Then $\sigma^2 = \mathrm{id}$ and $\sigma \neq \mathrm{id}$. Set

$$K := G \cap \mathrm{O}_d(\mathbb{R}),$$

which is a maximal compact subgroup of G. The differential of σ at the identity,

$$\Theta := d\sigma_e : \mathfrak{g} \to \mathfrak{g}, \quad x \mapsto -x^t,$$

is called the *Cartan involution*. The corresponding eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \qquad \Theta|_{\mathfrak{k}} = \mathrm{id}, \quad \Theta|_{\mathfrak{p}} = -\mathrm{id},$$

is known as the *Cartan decomposition*. It is a fact that $\mathfrak{k}$ is the Lie algebra of the maximal compact group K.

There is another class of important subalgebras for us is the maximal abelian subalgebras. Let $\mathfrak{a} \subseteq \mathfrak{g}$ be a maximal abelian subalgebra. By definition, any two elements of $\mathfrak{a}$ commute under the Lie bracket, and $\mathfrak{a}$ is maximal with this property. It carries an action of a reflection group whose description is omitted. The reflection hyperplanes divide $\mathfrak{a}$ into chambers; fix one such $\mathfrak{a}^+ \subseteq \mathfrak{a}$, called a Weyl chamber. Using the exponential map $\exp : \mathfrak{g} \to G$, since we are dealing matrix group,

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},\ x \in \mathfrak{g},$$

$A := \exp(\mathfrak{a})$ gives a connected abelian Lie subgroup of G; the chamber $\mathfrak{a}^+$ determines a semigroup $A^+ := \exp(\mathfrak{a}^+) \subseteq A$.

Example 2.1

Let $G = \mathrm{SL}_d(\mathbb{R}),\ K = \mathrm{SO}_d(\mathbb{R})$. Then

$$\mathfrak{g} := \mathfrak{sl}_d(\mathbb{R}) = \{a \in \mathrm{Mat}_d(\mathbb{R}) : \mathrm{Tr}(a) = 0\}$$

has decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \text{where } \mathfrak{k} = \mathfrak{so}_d(\mathbb{R}), \mathfrak{p} = \{a \in \mathfrak{g} : a = a^t\}.$$

We also observe that the bilinear form B on $\mathfrak{g}$ defined as

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},\ B(x,y) = \mathrm{Tr}(x \cdot y^t).$$

is *negative definite* on $\mathfrak{k}$ and *positive definite* on $\mathfrak{p}$.
A maximal abelian subalgebra for $\mathfrak{sl}_d(\mathbb{R})$ is given by

$$\mathfrak{a} := \left\{ \mathrm{diag}(t_1, \dots, t_d) : \sum_{i=1}^{d} t_i = 0 \right\} \subseteq \mathfrak{p}$$

The corresponding simply connected abelian subgroup is

$$A = \exp(\mathfrak{a}) = \left\{ \mathrm{diag}(e^{t_1}, \dots, e^{t_d}) : \sum_{i=1}^{d} t_i = 0 \right\} \subseteq G$$

A Weyl chamber of $\mathfrak{a}$ is given by

$$\mathfrak{a}^+ := \{\mathrm{diag}(t_1, \dots, t_d) \in \mathfrak{a} :\ t_1 \geq t_2 \cdots \geq t_d\} \subseteq \mathfrak{a}.$$

and the corresponding subsemigroup is $A^+ = \exp(\mathfrak{a}^+)$.

The symmetric space $X = G/K$

We now turn to the symmetric space $X := G/K$, endowed with the natural left G-action. Let $o := eK$ denote the basepoint. The space X enjoys the following properties:

- The canonical projection $\pi : G \to G/K$ induces, at the identity, a linear isomorphism

$$d\pi_e|_{\mathfrak{p}} : \mathfrak{p} \longrightarrow T_o(G/K).$$

- The space X carries a canonical G-invariant Riemannian metric, obtained from the restriction of the bilinear form B on $\mathfrak{p}$ defined above.

- Equipped with this metric, X is a complete, simply connected Riemannian manifold of non-positive sectional curvature. At the basepoint o, for unit vectors $a, b \in \mathfrak{p}$, the sectional curvature satisfies

$$K(a,b) = -\tfrac{1}{2}\|[a,b]\|^2.$$

- The following diagram commutes:

$$\begin{tikzcd}
\mathfrak{p} \arrow[r, "d\pi_e"] \arrow[d, "\exp"'] & T_o(G/K) \arrow[d, "\mathrm{Exp}_o"] \\
G \arrow[r, "\pi"'] & G/K
\end{tikzcd}$$

where Exp_o denotes the Riemannian exponential map. This expresses the fact that, in symmetric spaces, the Lie exponential map and the Riemannian exponential map coincide. Consequently, every geodesic ray in X issuing from o can be written as

$$\gamma(t) = \exp(ta) \cdot o, \quad a \in \mathfrak{p}.$$

The geodesic ray is unit speed if $\|a\|_o = 1$ with respect to the canonical metric on $\mathfrak{p}$.

The symmetric space $\mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R})$

We now examine in detail symmetric space, $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R})$, which serves as the principal example throughout our discussion. Let $\mathcal{P}^1(d)$ be the space of symmetric positive definite matrix with determinant 1.

Lemma 2.2

$\mathrm{SL}_d(\mathbb{R})$ acts on $\mathcal{P}^1(d)$ by

$$g \cdot x = gxg^t,\ g \in \mathrm{SL}_d(\mathbb{R}), x \in \mathcal{P}^1(d).$$

This action is transitive and

$$\mathrm{Stab}_{\mathrm{SL}_d(\mathbb{R})}(I_d) = \mathrm{SO}_d(\mathbb{R}).$$

As a result, we can identify $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R})$ with $\mathcal{P}^1(d)$ via

$$\Phi : \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R}) \longrightarrow \mathcal{P}^1(d),\ g\mathrm{SO}_d(\mathbb{R}) \mapsto (gg^t)^{\frac{1}{2}}.$$

Proof. **Exercise.** □

Remark 2.3 Every $x \in \mathcal{P}^1(d)$ has a unique square root in $\mathcal{P}^1(d)$. Since x is conjugate to a diagonal matrix via an orthogonal transformation:

$$x = k \cdot \mathrm{diag}(\lambda_1, \ldots, \lambda_d) \cdot k^t \text{ where } k \in \mathrm{SO}_d(\mathbb{R}) \text{ and } d_i > 0$$

Then $x^{1/2} := k \cdot \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_d}) \cdot k^t$ is the unique square root of x. Hence Φ in the lemma is well defined.

Let $I_d = \Phi(eK)$ be our base point. The tangent space at o can be identified as

$$T_o\mathcal{P}^1(d) \simeq \mathfrak{p} = \{A \in \mathfrak{sl}_d(\mathbb{R}) : A = A^t\}.$$

The canonical Riemannian metric on $\mathcal{P}^1(d)$ is defined by

$$\langle x, y\rangle_{I_d} = \mathrm{Tr}(xy^t) = \mathrm{Tr}(xy),\ x, y \in \mathfrak{p}.$$

This inner product extends to a $\mathrm{SL}_d(\mathbb{R})$-invariant Riemannian metric on all of $\mathcal{P}^1(d)$. With this metric, $\mathcal{P}^1(d)$ is a complete, simply connected, nonpositively curved Riemannian manifold.

At the basepoint I_d, for any $a \in \mathfrak{p}$, the curve

$$\gamma_a(t) = \Phi(\exp(ta)\cdot o) = (\exp(ta)\exp(ta)^t)^{\frac{1}{2}} = \exp(ta)$$

is a geodesic through I_d with speed $\|a\|_{I_d} = \sqrt{\mathrm{Tr}(a^2)} = \sum_{i=1}^d \lambda_i$, where $\lambda_1 \geq \cdots \geq \lambda_d > 0$ are the eigenvalues of a^2. Therefore, let $g \in \mathrm{SL}_d(\mathbb{R})$ and $\sigma_1^2 \geq \cdots \geq \sigma_d^2 > 0$ be the eigenvalues of $gg^t \in \mathcal{P}^1(d)$, then there exists some $k \in \mathrm{SO}_d(\mathbb{R})$ such that $x = (gg^t)^{\frac{1}{2}} = k\cdot\mathrm{diag}(\sigma_1, \dots, \sigma_d)\cdot k^t$. Let $a = \log(x) := k\cdot\mathrm{diag}(\log(\sigma_1), \dots, \log(\sigma_d))\cdot k^t \in \mathfrak{p}$, we have $x = \gamma_a(1) = \exp(a)$, which implies

$$d(I_d, x) = d(I_d, (gg^t)^{\frac{1}{2}}) = \left(\sum_{i=1}^d (\log(\sigma_i))^2\right)^{\frac{1}{2}} \asymp \log(\sigma_1) = \log\|g\|_{op}.$$

§3 Kaimanovich's characterization of regularity

We now turn to the discussion of Kaimanovich's characterization of regularity. The regularity of a sequence in a symmetric space of noncompact type can be characterized by rather simple conditions. The property of having non-positive curvature is crucial for this description to hold. Still, we will focus on the crucial example $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R}) \simeq \mathcal{P}^1(d)$.

Recall for $\mathrm{SL}_d(\mathbb{R})$, we have the Cartan (or KAK) decomposition:

$$g = k_1 a k_2,\ k_1, k_2 \in \mathrm{SO}_d(\mathbb{R}), a \in A^+$$

where A^+ is the image of $\mathfrak{a}^+ \subseteq \mathfrak{a}$ under exponential map. Then for $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R})$, all geodesic rays issuing from the basepoint $o = eK$ can be parametrized by $\gamma(t) = k\exp(t\alpha)\cdot o$ for some $k \in \mathrm{SO}_d(\mathbb{R}), \alpha \in \mathfrak{a}^+$. Moreover, given $x \in X$, there exists a **unique** $r(x) \in \mathfrak{a}^+$ such that

$$x = k\exp(r(x))\cdot o,\ \text{for some } k \in \mathrm{SO}_d(\mathbb{R}).$$

and $d(x, o) = \|r(x)\|_o$. $r(x)$ is called the *Cartan projection* of x. In fact, recall the calculation we did last subsection, we showed for $x = gK \in X$,

$$r(gK) = \mathrm{diag}(\log(\sigma_1), \dots, \log(\sigma_d)),$$

where $\sigma_1^2 \geq \cdots \geq \sigma_d^2 > 0$ are the eigenvalues of gg^t.

Theorem 3.1 (Kaimanovich)

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a symmetric space of noncompact type $X = G/K$ is regular if and only if the following two conditions are satisfied:

- **Small steps**: $d(x_n, x_{n+1}) = o(n)$;
- **Distances converge** $\frac{r(x_n)}{n} \to \alpha \in \mathfrak{a}^+$ as $n \to \infty$.

Before sketching the proof of this theorem, we discuss the main difference between the case of $d = 2$ and $d > 2$. Recall $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{g}$. Take $a_1, a_2 \in \mathfrak{a}$,

$$d((\exp(a_1)\exp(a_1)^t)^{\frac{1}{2}}, (\exp(a_2)\exp(a_2)^t)^{\frac{1}{2}}) = d(I_d, \exp(a_1)^{-\frac{1}{2}}\exp(a_2)\exp(a_1)^{-\frac{1}{2}})$$
$$\overset{*}{=} d(I_d, \exp(a_2 - a_1)) = \|a_2 - a_1\|_{I_d}.$$

In the step $*$, we used the fact $[a_1, a_2] = 0$ since $\mathfrak{a}$ is abelian. This implies $a \mapsto \exp(a)$ is an isometry and $A = \exp(A) \subseteq \mathcal{P}^1(d)$ is a totally geodesic submanifold isometric to $\mathbb{R}^{\dim \mathfrak{a}}$. A is called a *flat* in $X \simeq \mathcal{P}^1(d)$. Points in the flat behavior like in a Euclidean space, we do not have tools like hyperbolic sine law used in theorem 1.1 anymore. Once $\dim \mathfrak{a} \geq 2$, the example below shows that we can not expect that small steps will imply the angel converges anymore like the case in theorem 1.1.

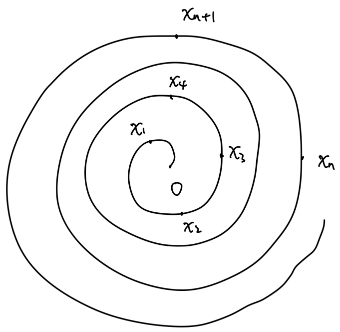

Figure 3: a sequence $\{x_n\}$ in $\mathbb{R}^2$ such that $d(x_n, x_{n+1}) = o(n)$ and $\lim \frac{\|x_n\|}{n}$ exists but the angel does converge

Sketch of proof of theorem 3.1. The necessity of the condition of the theorem is obvious. We establish its sufficiency. Without loss of generality one can assume that

$$x_n = k_n(\exp n\alpha) \cdot o, \; k_n \in K, \; \alpha \neq 0.$$

We must prove that from the small step condition $d(x_n, x_{n+1}) = o(n)$ the regularity of the sequence $\{x_n\}$ follows. The idea is the following: by the G-invariance of the metric,

$$d(x_n, x_{n+1}) = d(o, \exp(-n\alpha)k_n^{-1}k_{n+1}\exp((n+1)\alpha) \cdot o).$$

because the distance between successive points grows sublinearly, $k_n^{-1}k_{n+1}$ must eventually almost commute with the translation $\exp(\alpha)$. Geometrically, the small steps condition ensures that the sequence moves outward roughly along a single flat. As a consequence, this will give us a uniform upper bound for the negative curvature for the sequence geodesic plane $x_n o x_{n+1}$.

Let $\beta_n(t) = h_n(t)\exp a_n(t) \cdot o$ be a geodesic joining the points x_n and x_{n+1} with $\beta_n(0) = x_n$, $\beta_n(1) = x_{n+1}$. We denote $\dot{\beta}_n(t) \in \mathfrak{p}$ as the tangent vector to β_n at the point $\beta_n(t)$ translated along the geodesic connecting o and $\beta_n(t)$. Decompose $\dot{\beta}_n(t) = \dot{\xi}_n(t) + \dot{\eta}_n(t)$ with respect to $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$. Then $\dot{\xi}_n(t) = \dot{a}_n(t)$, $\dot{\eta}_n(t) \in \mathfrak{k}$ and

$$\|\dot{\beta}_n(t)\|^2 = \|\dot{\xi}_n(t)\|^2 + \|\dot{\eta}_n(t)\|^2.$$

Considering the submanifolds

$$R_n = \{h_n(t)\exp(s\alpha) \cdot o : t, s > 0\} \subseteq R = K\exp(\mathbb{R}_+\alpha) \cdot o.$$

Let d_n be the induced metric on R_n. One can show that $d_n(x_n, x_{n+1}) = o(n)$. It is easy to see that either x_n and x_{n+1} lie on the geodesic ray, issuing from o in which case R_n degenerates into a ray, or $\dot{\eta}_n(t) \neq 0$ for all t in which case R_n is the infinite sector included between the rays (o, x_n) and (o, x_{n+1}) and is geodesically convex (in the metric of R). In the latter case the curvature K_n of R_n at point $h_n(t)\exp(sa_n(t)) \cdot o$ is strictly negative. Furthermore, using some theory on root systems from Lie theory, it is a result of $d(x_n, x_{n+1}) = o(n)$ that

$$\sup_{n\in\mathbb{N}} K_n \leq \kappa < 0.$$

This uniform upper bound allows us to use Aleksandrov theorem on comparison of triangles to reduce the problem into what we have discussed in theorem 1.1.

□

§4 Symmetric spaces and the Oseledets theorem

The aim of this section is to build the bridge between what we has discussed and the Oseledets theorem. Let $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R}) \simeq \mathcal{P}^1(d)$. We begin with a definition

Definition 4.1 — The sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathrm{GL}_d(\mathbb{R})$ is called *Lyapunov regular* if the following conditions hold:

(i). the limit

$$\lim_{n\to\infty} \frac{1}{n} \log|\det A_n| = \lambda$$

exists;

(ii). there exist numbers

$$\lambda_1 > \lambda_2 > \cdots > \lambda_k$$

and a filtration

$$\mathbb{R}^d = V^1 \supset V^2 \supset \cdots \supset V^k \supset V^{k+1} = \{0\}$$

such that for every $i \in \{1, \ldots, k\}$,

$$\lim_{n\to\infty} \frac{1}{n} \log\|A_n v\| = \lambda_i$$

for $v \in V^i \setminus V^{i+1}$ and

$$\lambda = \sum_{i=1}^{k} (\dim V^i - \dim V^{i+1})\lambda_i.$$

Now we recall the Oseledets theorem using the notion of Lyapunov regularity.

Theorem 4.2
Let $(\Omega, \mathcal{B}, \mu, T)$ be an ergodic probability measure preserving system, and $A : \Omega \to \mathrm{GL}_d(\mathbb{R})$ a measurable linear cocycle over T satisfying the integrability condition

$$\int_\Omega \log^+ \|A(\omega)\|_{op}\, d\mu(\omega) < \infty.$$

Let

$$Z_\omega^{(n)} := A(T^{n-1}\omega)\cdots A(T\omega)A(\omega), \quad n \geq 1,$$

then for μ-a.e. $\omega \in \Omega$, the sequence $\{Z_\omega^{(n)}\}_{n\in\mathbb{N}}$ is Lyapunov regular. Moreover, the data $\lambda, \lambda_1 > \cdots > \lambda_k$ in the context of Lyapunov regularity is constant and $A(\omega)V_\omega^i = V_{T\omega}^i$ for μ-a.e. ω.

Remark 4.3 The first condition for sequence $\{Z_\omega^{(n)}\}_{n\in\mathbb{N}}$ to be Lyapunov regular just follows from the Birkhoff ergodic theorem applied to function $f(\omega) = |\det A(\omega)|$. Once we know this holds, we can always without loss of generality assume that our cocycle A takes values in $\mathrm{SL}_d(\mathbb{R})$ by following observation. Let $\tilde{A} : \Omega \to \mathrm{SL}_d(\mathbb{R})$, $\omega \mapsto \frac{1}{\det A(\omega)} A(\omega)$. Notice that

$$\tilde{Z}_\omega^{(n)} := \tilde{A}(T^{n-1}\omega)\cdots\tilde{A}(T\omega)\tilde{A}(\omega) = \frac{1}{\det Z_\omega^{(n)}} Z_\omega^{(n)}, \quad n \geq 1,$$

If there exist $\tilde{\lambda}_1 > \tilde{\lambda}_2 > \cdots > \tilde{\lambda}_k$ and

$$\mathbb{R}^d = V_\omega^1 \supset V_\omega^2 \supset \cdots \supset V_\omega^k \supset V_\omega^{k+1} = \{0\}$$

that satisfy 2,3 for $\tilde{A}$. Then $\tilde{\lambda}_1 + \lambda > \tilde{\lambda}_2 + \lambda > \cdots > \tilde{\lambda}_k + \lambda$ together with the same filtration would satisfy 2,3 for A.

Proposition 4.4
Let $\{A_n\}_{n\in\mathbb{N}} \subset \mathrm{SL}_d(\mathbb{R})$, then the following conditions are equivalent:

(i). the sequence $\{A_n\}_{n\in\mathbb{N}}$ is Lyapunov regular;

(ii). there exists a symmetric positive definite matrix $\Lambda \in \mathcal{P}^1(d)$ such that

$$\log \|A_n \Lambda^{-n}\|_{op} = o(n).$$

(iii). $\log \|A_{n+1}A_n^{-1}\|_{op} = o(n)$ and

$$\Lambda = \lim_{n\to\infty} ((A_n)^{-1}(A_n^{-1})^t)^{\frac{1}{2n}}$$

exists.

(iv). $x_n = ((A_n)^{-1}(A_n^{-1})^t)^{1/2}$ is regular in $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R}) \simeq \mathcal{P}^1(d)$.

Proof. The equivalence of the (ii), (iii), (iv) is exact the content of Kamanovich's theorem 3.1. We prove (i)⇔(ii) here. Assume first (i). Given Oseledets filtration

$$\mathbb{R}^d = V^1 \supset V^2 \supset \cdots \supset V^k \supset V^{k+1} = \{0\},$$

define W_i be the orthogonal complement of V^{i+1} in V^i, $1 \leq i \leq k$. Notice that $W_k = V^k$. Then by induction, one can easily show that

$$\mathbb{R}^d = W_1 \oplus W_2 \oplus \cdots \oplus W_k.$$

Now define a symmetric positive definite matrix Λ by requiring that $\Lambda w = e^{\lambda_i} w$ for $w \in W_i$, $1 \leq i \leq k$. Then for $v \in W_i \subset V^i \setminus V^{i+1}$, $\log \|A_n \Lambda^{-n} v\| = o(n)$, as stated.

Conversely, given $\Lambda \in \mathcal{P}^1(d)$, let $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ be its eigenvalues. Define W_i to be the eigenspaces of Λ corresponding to the eigenvalue λ_i and $V^i = W_i + \cdots + W_k$, $1 \leq i \leq k$. Then we claim that

$$\mathbb{R}^d = V^1 \supset V^2 \supset \cdots \supset V^k \supset V^{k+1} = \{0\}$$

is the filtration we want. This is because for $v \in V^i \setminus V^{i+1}$

$$\log \|A_n v\| = \log \|\lambda_i^n A_n \Lambda^{-n} v\| = n\lambda_i + o(n).$$

$\square$

We end the proof with the statement of geometric form of mulplicative ergodic theorm, whose proof is the same as the proof of theorem 1.1.

Theorem 4.5

Let $(\Omega, \mathcal{B}, \mu, T)$ be an ergodic probability measure preserving system, and $A : \Omega \to \mathrm{SL}_d(\mathbb{R})$ a measurable linear cocycle over T satisfying the integrability conditions

$$\int_\Omega \log^+ \|A(\omega)\|_{op} \, d\mu(\omega) < \infty.$$

For each $\omega \in \Omega$, let

$$A_\omega^{(n)} := A(\omega)A(T\omega)\cdots A(T^{n-1}\omega), \quad n \geq 1.$$

Then for almost every $\omega \in \Omega$, the sequence $\{A_\omega^{(n)} \cdot o\}_{n\in\mathbb{N}}$ in $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SO}_d(\mathbb{R})$ is regular.

Through proposition 4.4, after adjusting the integrability conditions and multiplication order, one can see theorem 4.2 and theorem 4.5 are nearly the same.